CONSTRAINTS FROM SPECTRUM OF SCALAR FIELDS IN THE 3-3-1 MODEL WITH CKM MECHANISM

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Abstract. We explore constraints from positivity of scalar mass spectra in the 3-3-1 model with CKS mechanism. The conditions for positivity of the diagonal elements are most important since other constraints are followed from the first ones.

Keywords: extensions of electroweak gauge sector, extensions of electroweak Higgs sector, electroweak radiative corrections.

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I. INTRODUCTION

It is well known that the Higgs mechanism plays a very important role for production of particle masses. In general, the Higgs potential has to be bounded from below to ensure its stability [1]. In the Standard Model (SM) it is enough to have a positive Higgs boson quartic coupling \( \lambda > 0 \). In the extended models with more scalar fields, the potential should be bounded from below in all directions in the field space as the field strength approaches infinity. It is interesting to note that the square scalar mass matrix is associated with the Hessian matrix \( H_{ij} \) determined at the vacuum

\[
(H_0)_{ij} = \left. \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \right|_{\phi = \text{min}}.
\]
The condition for the potential to be bounded from below also leads to positivity of the above matrix [2]. The mentioned condition practically is the positivity of the principal minors. In this paper we focus our attention on positivity of scalar mass spectra and intend to get the constraints from it.

Let us remind the useful definition. A symmetric matrix $M^2$ of quadric form $x^T M^2 x$ for all vector $x$ in $\mathbb{R}^n$ with the following properties

$$\begin{cases} x^T M^2 x \geq 0, & M^2 \text{ is called positive semidefinite,} \\ x^T M^2 x > 0, & M^2 \text{ is called positive definite.} \end{cases}$$

If $M^2$ is $2 \times 2$ matrix with elements being $M^2_{ij}, i, j = 1, 2$ then Eq.(2) leads to the following conditions

$$M^2_{11} > 0, \quad M^2_{22} > 0,$$

$$M^2_{12} + \sqrt{M^2_{11} M^2_{22}} > 0.$$  

For $3 \times 3$ matrix we have [1]

$$M^2_{11} > 0, \quad M^2_{22} > 0, \quad M^2_{33} > 0,$$

$$M^2_{12} + \sqrt{M^2_{11} M^2_{22}} > 0,$$

$$M^2_{13} + \sqrt{M^2_{11} M^2_{33}} > 0,$$

$$M^2_{23} + \sqrt{M^2_{22} M^2_{33}} > 0,$$

and

$$\sqrt{M^2_{11} M^2_{22} M^2_{33} + M^2_{12} \sqrt{M^2_{33} + M^2_{13} \sqrt{M^2_{22} + M^2_{23}}} \sqrt{M^2_{11}}} > 0,$$  

$$det M^2 = M^2_{11} M^2_{22} M^2_{33} - (M^2_{12} M^2_{33} + M^2_{13} M^2_{22} + M^2_{23} M^2_{11}) + 2 M^2_{12} M^2_{13} M^2_{23} > 0.$$  

For the matrices of rank 4 or 5, the reader is referred to Refs. [3, 4].

One of the main purposes of the models based on the gauge group $SU(3)_C \times SU(3)_L \times U(1)_X$ (for short, 3-3-1 model) [5, 6] is concerned with the search of an explanation for the number of generations of fermions. Combined with the QCD asymptotic freedom, the 3-3-1 models provide an explanation for the number of fermion generations. To provide an explanation for the observed pattern of SM fermion masses and mixings, various 3-3-1 models with flavor symmetries [7–9] and radiative seesaw mechanisms [7, 12] have been proposed in the literature. However, some of them involve non-renormalizable interactions [10], others are renormalizable but do not address the observed pattern of fermion masses and mixings due to the unexplained huge hierarchy among the Yukawa couplings [8] and others are only focused either in the quark mass hierarchy [8, 11], or in the study of the neutrino sector [12, 13], or only include the description of SM fermion mass hierarchy, without addressing the mixings in the fermion sector [14].

It is interesting to find an alternative explanation for the observed SM fermion mass and mixing pattern. The first renormalizable extension of the 3-3-1 model with $\beta = -\frac{1}{\sqrt{3}}$, which explains the SM fermion mass hierarchy by a sequential loop suppression has been done in Ref. [15]. This model is called by the 3-3-1 model with Carcamo-Kovalenko-Schmidt (CKS) mechanism.
The aim of this paper is to apply the procedure in (2) for the recently proposed 3-3-1 model with CKS mechanism.

The further content of this paper is as follows. In Sect. II, we briefly present particle content of scalar sector and spontaneous symmetry breaking (SSB) of the model. The Higgs sector is considered in Sect. III. The Higgs sector consists of two parts: the first part contains lepton number conserving terms and the second one is lepton number violating. We study in details the first part and show that the Higgs sector has all necessary ingredients. We make conclusions in Sect. IV.

II. SCALAR FIELDS OF THE MODEL

In the model under consideration, the Higgs sector contains three scalar triplets: $\chi$, $\eta$ and $\rho$ and seven singlets $\phi_1^0$, $\phi_2^0$, $\xi^0$, $\phi_1^+$, $\phi_2^+$, $\phi_3^+$ and $\phi_4^+$. Hence, the scalar spectrum of the model is composed of the following fields

$$
\chi = \langle \chi \rangle + \chi' \sim (1, 3, -1/3),
$$

$$\langle \chi \rangle = \left(0, 0, \frac{v_\chi}{\sqrt{2}}\right)^T, \quad \chi' = \left(\chi_1^0, \chi_2^-, \frac{1}{\sqrt{2}}(R_{\chi_3^0} - iL_{\chi_3^0})\right)^T,
$$

$$
\rho = \left(\rho_1^+, \frac{1}{\sqrt{2}}(R_{\rho} - iL_{\rho}), \rho_3^+\right)^T \sim (1, 3, 2/3),
$$

$$
\eta = \langle \eta \rangle + \eta' \sim (1, 3, -1/3),
$$

$$\langle \eta \rangle = \left(\frac{v_\eta}{\sqrt{2}}, 0, 0\right)^T, \quad \eta' = \left(\frac{1}{\sqrt{2}}(R_{\eta_1^0} - iL_{\eta_1^0}), \eta_2^-, \eta_3^0\right)^T,
$$

$$
\phi_1^0 \sim (1, 1, 0), \quad \phi_2^0 \sim (1, 1, 0),
$$

$$
\phi_1^+ \sim (1, 1, 1), \quad \phi_2^+ \sim (1, 1, 1), \quad \phi_3^+ \sim (1, 1, 1), \quad \phi_4^+ \sim (1, 1, 1),
$$

$$
\xi^0 = \langle \xi^0 \rangle + \xi^0', \quad \langle \xi^0 \rangle = \frac{v_\xi}{\sqrt{2}}, \quad \xi^0' = \frac{1}{\sqrt{2}}(R_{\xi_0^0} - iL_{\xi_0^0}) \sim (1, 1, 0). \tag{12}
$$

The $Z_4 \times Z_2$ assignments of the scalar fields are shown in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>$\chi$</th>
<th>$\eta$</th>
<th>$\rho$</th>
<th>$\phi_1^0$</th>
<th>$\phi_2^0$</th>
<th>$\phi_1^+$</th>
<th>$\phi_2^+$</th>
<th>$\phi_3^+$</th>
<th>$\phi_4^+$</th>
<th>$\xi^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_4$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>$i$</td>
<td>$i$</td>
<td>-1</td>
<td>-1</td>
<td>$1$</td>
</tr>
<tr>
<td>$Z_2$</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$1$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>$1$</td>
</tr>
</tbody>
</table>

The fields with nonzero lepton number are presented in Table 2. Note that the three gauge singlet neutral leptons $N_{iR}$ as well as the elements in the third component of the lepton triplets, namely $\nu_{iL}$ have lepton number equal to $-1$. 

The further content of this paper is as follows. In Sect. II, we briefly present particle content of scalar sector and spontaneous symmetry breaking (SSB) of the model. The Higgs sector is considered in Sect. III. The Higgs sector consists of two parts: the first part contains lepton number conserving terms and the second one is lepton number violating. We study in details the first part and show that the Higgs sector has all necessary ingredients. We make conclusions in Sect. IV.
Table 2. Nonzero lepton number $L$ of fields

<table>
<thead>
<tr>
<th>$T_L$</th>
<th>$J_{1L}$</th>
<th>$J_{2L}$</th>
<th>$v_L$</th>
<th>$e_{1L}$</th>
<th>$E_{1L}$</th>
<th>$N_{1R}$</th>
<th>$\Psi_R$</th>
<th>$\chi_1^0$</th>
<th>$\chi_2^0$</th>
<th>$\eta^0_1$</th>
<th>$\rho_1^0$</th>
<th>$\phi_1^0$</th>
<th>$\phi_2^0$</th>
<th>$\phi_3^0$</th>
<th>$\xi^0_1$</th>
<th>$i = 1, 2, 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>$-2$</td>
<td>$2$</td>
<td>$2$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$2$</td>
<td>$2$</td>
<td>$-2$</td>
<td>$-2$</td>
<td>$-2$</td>
<td>$-2$</td>
<td>$-2$</td>
<td>$-2$</td>
</tr>
</tbody>
</table>

III. THE SCALAR POTENTIAL

The renormalizable potential contains three parts [16]. The first part is given by

$$V_{LNC} = \mu_\chi^2 \chi^\dagger \chi + \mu_\rho^2 \rho + \mu_\eta^2 \eta + \sum_{i=1}^4 \mu_{\phi_i}^2 \phi_i^+ \phi_i^- + \sum_{i=1}^2 \mu_{\phi_i}^2 \phi_i^0 \phi_i^{0*} + \mu_\phi^2 \phi_0^0 \phi_0^{0*}$$

$$+ \chi^\dagger (\lambda_{13} \chi^\dagger \chi + \lambda_{18} \rho^+ \rho + \lambda_{35} \eta^+ \eta) + \rho^+ \rho (\lambda_{14} \rho^+ \rho + \lambda_{56} \eta^+ \eta) + \lambda_{17} (\eta^+ \eta)^2$$

$$+ \lambda_7 (\chi^\dagger \rho) (\rho^+ \chi) + \lambda_8 (\chi^\dagger \eta) (\eta^+ \chi) + \lambda_9 (\rho^+ \eta) (\eta^+ \rho)$$

$$+ \chi^\dagger (\sum_{i=1}^4 \lambda_i \chi \phi_i^+ \phi_i^- + \sum_{i=1}^2 \lambda_i \phi^0 \phi_i^0 + \lambda_4 \phi_0^0 \phi_0^{0*} + \lambda_5 \phi_0^0 \phi_0^{0*}$$

$$+ \sum_{i=1}^4 \phi_i^+ \phi_i^- \left( \sum_{j=1}^4 \lambda_{ij} \phi_j^+ \phi_j^- + \sum_{j=1}^2 \lambda_{ij} \phi_j^0 \phi_j^{0*} + \lambda_{ij} \phi_0^0 \phi_0^{0*} \right)$$

$$+ \sum_{j=1}^2 \phi_j^0 \phi_j^{0*} \left( \sum_{j=1}^2 \lambda_{ij} \phi_j^0 \phi_j^{0*} + \lambda_4 \phi_0^0 \phi_0^{0*} \right)$$

$$+ \left\{ \lambda_{10} (\phi_3^+)^2 (\phi_3^-)^2 + \lambda_{11} (\phi_4^+)^2 (\phi_4^-)^2 + \lambda_{12} (\phi_5^+)^2 (\phi_5^-)^2 \right\}$$

$$+ w_1 (\phi_0^0)^2 \phi_0^0 + w_2 \chi^\dagger \rho \phi_3^- + w_3 \eta^+ \chi \phi_0^0 + w_4 (\phi_0^0)^2 \phi_0^0 + w_5 \phi_3^+ \phi_4^- \phi_1^0 + w_6 \phi_4^+ \phi_4^- \phi_1^0$$

$$+ \chi \rho \eta (\lambda_{14} \phi_0^0 + \lambda_3 \phi_1^0 + \lambda_{16} \phi_1^0) + \chi \rho \phi_4 (\lambda_{15} \phi_0^0 + \lambda_{16} \phi_1^0) + \lambda_3 \eta^+ \rho \phi_3^- \phi_0^0 + \lambda_4 \phi_1^0 \phi_2^- \phi_0^0$$

$$+ \lambda_9 \phi_3^- \phi_4^+ + \lambda_{20} \phi_0^+ \phi_4^- (\phi_0^0)^2 + \lambda_{21} (\phi_0^0)^3 \phi_0^0$$

$$+ \lambda_{22} \chi^\dagger \chi + \lambda_{23} \rho^+ \rho + \lambda_{24} \eta^+ \eta + \sum_{i=1}^4 \lambda_{61} \phi_i^+ \phi_i^- + \sum_{i=1}^2 \lambda_{62} \phi_0^0 \phi_0^{0*}$$

$$+ \lambda_{25} \phi_0^0 \phi_0^{0*} (\phi_0^0)^2 + h.c. \right\}$$

The second part is a lepton number violating one (the subgroup $U(1)_{L_\nu}$ is violated) and the third breaking softly $Z_4 \times Z_2$ are given in Ref. [16].
Expanding the Higgs potential around VEVs, one gets the constraint conditions at the tree level as follows

\[ w_3 = 0, \]
\[ -\mu^2_x = v_x^2 \lambda_{13} + \frac{1}{2} v_\eta^2 \lambda_5 + \frac{1}{2} \lambda_\xi v_\xi^2, \]
\[ -\mu^2_\eta = v_\eta^2 \lambda_{17} + \frac{1}{2} v_x^2 \lambda_5 + \frac{1}{2} \lambda_\xi v_\xi^2, \]
\[ -\mu^2_\xi = \frac{1}{2} \lambda_\xi v_\xi^2 + \frac{1}{2} \lambda_\xi v_\xi^2. \]

Applying the constraint conditions in (14), the charged scalar sector contains two massless fields: \( \eta^+_2 \) and \( \chi^+_2 \) which are Goldstone bosons eaten by the \( W^\pm \) and \( Y^+ \) gauge bosons, respectively. The other massive fields are \( \phi^+_1, \phi^+_2 \) and \( \phi^+_3 \) with respective masses

\[ m^2_{\phi^+_1} = \mu^2_\phi^+_1 + \frac{1}{2} \left[ v_x^2 \lambda_{13} \chi^\phi + v_\eta^2 \lambda_5 \eta^\phi + v_\xi^2 \lambda_4 \xi^\phi \right], \]
\[ m^2_{\phi^+_2} = \mu^2_\phi^+_2 + \frac{1}{2} \left[ v_x^2 \lambda_{17} \chi^\phi + v_\eta^2 \lambda_5 \eta^\phi + v_\xi^2 \lambda_4 \xi^\phi \right], \]
\[ m^2_{\phi^+_3} = \mu^2_\phi^+_3 + \frac{1}{2} \left[ v_x^2 \lambda_{19} \chi^\phi + v_\eta^2 \lambda_5 \eta^\phi + v_\xi^2 \lambda_4 \xi^\phi \right]. \]  

In addition, in the basis \( (\rho^+_1, \rho^+_2, \phi^+_3) \), there is the mass mixing matrix

\[ M^2_{\text{charged}} = \begin{pmatrix} A + \frac{1}{2} v_\eta^2 (\lambda_6 + \lambda_9) & 0 & \frac{1}{2} v_\eta v_\xi \lambda_3 \\ 0 & A + \frac{1}{2} \left( v_x^2 \lambda_7 + v_\eta^2 \lambda_6 \right) & \frac{1}{\sqrt{2}} v_x w_2 \\ \frac{1}{2} v_\eta v_\xi \lambda_3 & \frac{1}{\sqrt{2}} v_x w_2 & \mu^2_{\phi^+_3} + B_3 \end{pmatrix}, \]  

where we have used the following notations

\[ A \equiv \mu^2_\rho + \frac{1}{2} \left[ v_x^2 \lambda_{18} + \lambda_\rho v_\xi^2 \right], \]
\[ B_i \equiv \frac{1}{2} \left( v_x^2 \lambda_{i} \chi^\phi + v_\eta^2 \lambda_5 \eta^\phi + v_\xi^2 \lambda_4 \xi^\phi \right), \quad i = 1, 2, 3, 4. \]  

The conditions in Eqs. (4 - 7) yield

\[ A + \frac{1}{2} v_\eta^2 (\lambda_6 + \lambda_9) > 0, \]
\[ A + \frac{1}{2} \left( v_x^2 \lambda_7 + v_\eta^2 \lambda_6 \right) > 0, \]
\[ \mu^2_{\phi^+_i} + B_3 > 0, \]
\[ \sqrt{A + \frac{1}{2} v_\eta^2 (\lambda_6 + \lambda_9)} \left( A + \frac{1}{2} \left( v_x^2 \lambda_7 + v_\eta^2 \lambda_6 \right) \right) > 0, \]
\[ \frac{1}{2} v_\eta v_\xi \lambda_3 + \sqrt{A + \frac{1}{2} v_\eta^2 (\lambda_6 + \lambda_9)} \left( \mu^2_{\phi^+_3} + B_3 \right) > 0, \]
\[ \frac{1}{\sqrt{2}} v_x w_2 + \sqrt{A + \frac{1}{2} \left( v_x^2 \lambda_7 + v_\eta^2 \lambda_6 \right)} \left( \mu^2_{\phi^+_3} + B_3 \right) > 0. \]
Note that in this case the constraints in (19) are just enough or other word speaking, if the conditions of semi-definition for diagonal elements are fulfilled then other ones are automatically satisfied.

Now we turn into CP-odd Higgs sector. There are three massless fields: $I_\chi, I_\eta$ and $I_\xi$. The field $I_\phi$ has the following squared mass

$$m_{I_\phi}^2 = \mu^2_\phi + B'_2,$$

where

$$B'_n \equiv \frac{1}{2} \left( v_\chi^2 \lambda_n \phi^2 + v_\eta^2 \lambda_n \phi^2 + v_\xi^2 \lambda_n \phi^2 \right), \quad n = 1, 2. \quad (21)$$

There are other two mass matrices as follows: Firstly, in the basis $(I_\chi_0, I_\eta_0)$, the matrix is

$$m_{\text{CP odd}1}^2 = \frac{\lambda_8}{2} \begin{pmatrix} v_\eta^2 & -v_\chi v_\eta \\ -v_\chi v_\eta & v_\chi^2 \end{pmatrix}. \quad (22)$$

The matrix in (23) provides two physical states

$$G_1 = \cos \theta_a I_\chi_0 + \sin \theta_a I_\eta_0,$$

$$A_1 = -\sin \theta_a I_\chi_0 + \cos \theta_a I_\eta_0, \quad (24)$$

where

$$\tan \theta_a = \frac{v_\eta}{v_\chi}. \quad (25)$$

The field $G_1$ is massless while the field $A_1$ has mass as follows

$$m_{A_1}^2 = \frac{\lambda_8 v_\chi^2}{2 \cos^2 \theta_a}. \quad (26)$$

Secondly, in the basis $(I_\phi_1, I_\rho)$, the matrix is

$$m_{\text{CP odd}2}^2 = \begin{pmatrix} \mu^2_\phi - C + B_1 & \frac{1}{2} v_\chi v_\eta (\lambda_1 - \lambda_2) \\ \frac{1}{2} v_\chi v_\eta (\lambda_1 - \lambda_2) & A + \frac{\lambda_6}{2} v_\eta^2 \end{pmatrix}, \quad (27)$$

where we have denoted

$$C \equiv v_\chi^2 \lambda_{22} + v_\eta^2 \lambda_{24} + v_\xi^2 \lambda_{25} \quad (28)$$

The conditions in (4) yield

$$\mu^2_\phi - C + B_1 > 0, \quad A + \frac{\lambda_6}{2} v_\eta^2 > 0, \quad (29)$$

$$\frac{1}{2} v_\chi v_\eta (\lambda_1 - \lambda_2) + \sqrt{(\mu^2_\phi - C + B_1) \left( A + \frac{\lambda_6}{2} v_\eta^2 \right)} > 0. \quad (30)$$

The above conditions provide the following constraints:

i) If $\lambda_1 < \lambda_2$, then

$$\left( \mu^2_\phi - C + B_1 \right) \left( A + \frac{\lambda_6}{2} v_\eta^2 \right) > \frac{v_\chi^2 v_\eta^2}{4} (\lambda_1 - \lambda_2)^2.$$

ii) If $\lambda_1 > \lambda_2$, there are only conditions given in (30).
Generally, physical states of matrix (27) are
\[
\begin{pmatrix}
A_2 \\
A_3
\end{pmatrix} = \begin{pmatrix}
\cos \theta_\rho & \sin \theta_\rho \\
-\sin \theta_\rho & \cos \theta_\rho
\end{pmatrix}
\begin{pmatrix}
I_{\phi_1} \\
I_\rho
\end{pmatrix},
\]
(31)
where the mixing angle is given by
\[
\tan 2\theta_\rho = \frac{v_\chi v_\eta (\lambda_1 - \lambda_2)}{\mu_{\phi_1}^2 - C + B_1 - A - \frac{\lambda_6}{2} v_\eta^2},
\]
(32)
and their squared masses as follows
\[
m_{\lambda_2}^2 = \frac{1}{2} \left\{ A + D_1 - \sqrt{(A - D_1)^2 + v_\eta^2 \left[ 2(A - D_1) \lambda_6 + v_\eta^2 \lambda_6^2 + v_\chi^2 (\lambda_{13} - \lambda_{14})^2 \right]} \right\},
\]
\[
m_{\lambda_3}^2 = \frac{1}{2} \left\{ A + D_1 + \sqrt{(A - D_1)^2 + v_\eta^2 \left[ 2(A - D_1) \lambda_6 + v_\eta^2 \lambda_6^2 + v_\chi^2 (\lambda_{13} - \lambda_{14})^2 \right]} \right\},
\]
(33)
where
\[
D_1 = \mu_{\phi_1}^2 + B_1 - C + \frac{1}{2} v_\eta^2 \lambda_6.
\]
Next, the CP-even scalar sector is our task. Ones have one massive field, namely \( R_{\phi_2} \) with mass
\[
m_{R_{\phi_2}}^2 = m_{\phi_2}^2 = \mu_{\phi_2}^2 + B_2'
\]
\[
= \mu_{\phi_2}^2 + \frac{1}{2} \left( v_\chi^2 \lambda_2^2 \phi + v_\eta^2 \lambda_2^2 \phi^2 + v_\chi^2 \lambda_2^2 \phi^3 \right).
\]
(35)
As mentioned in Ref. [15], the lightest scalar \( \phi_0^2 \) is possible DM candidate. Therefore from (35), the following condition is reasonable
\[
\mu_{\phi_2}^2 = -\frac{1}{2} \left( v_\chi^2 \lambda_2^2 \phi + v_\eta^2 \lambda_2^2 \phi^2 \right).
\]
(36)
In this case, the model contains the complex scalar DM \( \phi_2^0 \) with mass
\[
m_{R_{\phi_2}}^2 = m_{\phi_2}^2 = \frac{1}{2} v_\eta^2 \lambda_2^2 \eta \phi.
\]
(37)
Other three mass matrices are
iii) In the basis \((R_{\chi_1^0}, R_{\eta_1^0})\), the matrix is
\[
m_{\text{CP-even}}^2 = \frac{\lambda_8}{2} \begin{pmatrix}
v_\eta^2 & v_\chi v_\eta \\
v_\chi v_\eta & v_\chi^2
\end{pmatrix}.
\]
(38)
This matrix is completely similar to that in (23). Thus, two physical states are
\[
R_{G_1} = \cos \theta_\alpha R_{\chi_1^0} + \sin \theta_\alpha R_{\eta_1^0},
\]
\[
H_1 = -\sin \theta_\alpha R_{\chi_1^0} + \cos \theta_\alpha R_{\eta_1^0},
\]
(39)
where \( R_{G_1} \) is massless while the field \( H_2 \) has mass as follows
\[
m_{H_1}^2 = m_{\lambda_1}^2 = \frac{\lambda_8 v_\chi^2}{2 \cos^2 \theta_\alpha}.
\]
(40)
iV) In the basis \((R_\rho, R_{\varphi_1})\), the matrix is
\[
m^2_{C,\text{even}} = \begin{pmatrix}
A + \frac{\lambda_6 v_\eta^2}{2} & -\frac{1}{2} v_\chi v_\eta (\lambda_1 + \lambda_2) \\
-\frac{1}{2} v_\chi v_\eta (\lambda_1 + \lambda_2) & \frac{\mu^2_{\varphi_1}}{2} + C + B_1
\end{pmatrix}.
\] (41)

As before, ones get
\[
A + \frac{\lambda_6 v_\eta^2}{2} > 0, \quad \mu^2_{\varphi_1} + C + B_1 > 0, \quad (42)
\]
\[
-\frac{1}{2} v_\chi v_\eta (\lambda_1 + \lambda_2) + \sqrt{(A + \frac{\lambda_6 v_\eta^2}{2}) (\mu^2_{\varphi_1} + C + B_1)} > 0.
\] (43)

Thus, if \(\lambda_1 + \lambda_2 > 0\), then
\[
(A + \frac{\lambda_6 v_\eta^2}{2}) (\mu^2_{\varphi_1} + C + B_1) > \frac{v_\chi^2 v_\eta^2}{4} (\lambda_6 + \lambda_2)^2.
\] (44)

If \(\lambda_1 + \lambda_2 \leq 0\), there are only conditions in (42).

The physical states of matrix (41) are
\[
\begin{pmatrix}
H_2 \\
H_3
\end{pmatrix} = \begin{pmatrix}
\cos \theta_r & \sin \theta_r \\
-\sin \theta_r & \cos \theta_r
\end{pmatrix}
\begin{pmatrix}
R_\rho \\
R_{\varphi_1}
\end{pmatrix},
\] (45)

where the mixing angle is given by
\[
\tan 2\theta_r = \frac{v_\chi v_\eta (\lambda_1 + \lambda_2)}{(\mu^2_{\varphi_1} + C + B_1 - A - \frac{\lambda_6 v_\eta^2}{2})},
\] (46)

and their squared masses are identified by
\[
m^2_{H_2} = \frac{1}{2} \left\{ A + D_2 - \sqrt{(A - D_2)^2 + v_\eta^2 \left[ 2(A - D_2) \lambda_6 + v_\eta^2 \lambda_6^2 + v_\chi^2 (\lambda_13 + \lambda_14)^2 \right]} \right\},
\]
\[
m^2_{H_3} = \frac{1}{2} \left\{ A + D_2 + \sqrt{(A - D_2)^2 + v_\eta^2 \left[ 2(A - D_2) \lambda_6 + v_\eta^2 \lambda_6^2 + v_\chi^2 (\lambda_13 + \lambda_14)^2 \right]} \right\},
\] (47)

where
\[
D_2 = \mu^2_{\varphi_1} + B_1 + C + \frac{1}{2} v_\eta^2 \lambda_6.
\] (48)

v) In the basis \((R_\chi, R_\eta, R_\xi)\), the matrix is
\[
m^2_{C,\text{even}} = \begin{pmatrix}
2 v_\chi^2 \lambda_13 & v_\chi v_\eta \lambda_5 & \lambda_\xi v_\chi v_\xi \\
v_\chi v_\eta \lambda_5 & 2 v_\eta^2 \lambda_17 & \lambda_\xi v_\eta v_\xi \\
\lambda_\xi v_\chi v_\xi & \lambda_\xi v_\eta v_\xi & 2 \lambda_\xi v_\eta^2
\end{pmatrix}.
\] (49)
Again, in this case the constraints in Eqs (4 - 9) are given by
\[
\lambda_{13} > 0, \lambda_{17} > 0, \lambda_{\xi} > 0,
\]

\[
v_{\chi} v_\eta \lambda_5 + \sqrt{(2v_{\chi}^2 \lambda_{13}) (2v_\eta^2 \lambda_{17})} > 0 \Rightarrow \lambda_5 > -2\sqrt{(\lambda_{13} \lambda_{17})},
\]

\[
\lambda_{\chi \xi} v_{\chi} v_\xi + \sqrt{(2v_{\chi}^2 \lambda_{13}) (2\lambda_{\xi} v_\xi^2)} > 0 \Rightarrow \lambda_{\chi \xi} > -2\sqrt{(\lambda_{13} \lambda_{\xi})},
\]

\[
\lambda_{\eta \xi} v_\eta v_\xi + \sqrt{(2v_\eta^2 \lambda_{17}) (2\lambda_{\xi} v_\xi^2)} > 0 \Rightarrow \lambda_{\eta \xi} > -2\sqrt{(\lambda_{17} \lambda_{\xi})},
\]

\[
\left(2v_{\chi}^2 \lambda_{13}\right) \left(2v_\eta^2 \lambda_{17}\right) \left(2\lambda_{\xi} v_\xi^2\right) + (v_{\chi} v_\eta \lambda_5)^2 \left(2\lambda_{\xi} v_\xi^2\right) + (\lambda_{\chi \xi} v_{\chi} v_\xi)^2 \left(2v_\eta^2 \lambda_{17}\right)
\]

\[
+ \lambda_{\eta \xi} v_\eta v_\xi \sqrt{(2v_{\chi}^2 \lambda_{13})} > 0
\]

\[
\Rightarrow 2\sqrt{\lambda_{13} \lambda_{17} \lambda_{\xi}} + \sqrt{\lambda_5} + \lambda_{\chi \xi} \sqrt{\lambda_{17}} + \lambda_{\eta \xi} \sqrt{\lambda_{13}} > 0,
\]

\[
\left(2v_{\chi}^2 \lambda_{13}\right) \left(2v_\eta^2 \lambda_{17}\right) \left(2\lambda_{\xi} v_\xi^2\right) - [(v_{\chi} v_\eta \lambda_5)^2 \left(2\lambda_{\xi} v_\xi^2\right)] + (\lambda_{\chi \xi} v_{\chi} v_\xi)^2 \left(2v_\eta^2 \lambda_{17}\right)
\]

\[
+ \left(2v_{\chi}^2 \lambda_{13}\right) (\lambda_{\eta \xi} v_\eta v_\xi)^2 + 2v_{\chi} v_\eta \lambda_5 \lambda_{\chi \xi} v_{\chi} v_\xi \lambda_{\eta \xi} v_\eta v_\xi > 0
\]

\[
\Rightarrow 4\lambda_{13} \lambda_{17} \lambda_{\xi} - [(\lambda_5)^2 \lambda_{\xi} + (\lambda_{\chi \xi})^2 \lambda_{17} + (\lambda_{\eta \xi})^2 \lambda_{13}] + \lambda_5 \lambda_{\chi \xi} \lambda_{\eta \xi} > 0.
\]

### III.1. Special cases

To find solutions in Higgs sector, we should make some simplifications.

#### III.1.1. The SM-like Higgs boson

We consider now the matrix (49): with the basis \( (R_{\chi}, R_{\eta}, R_{\xi}) \)

\[
m_{\text{Peven3}}^2 = \begin{pmatrix}
2v_{\chi}^2 \lambda_{13} & v_{\chi} v_\eta \lambda_5 & \lambda_{\chi \xi} v_{\chi} v_\xi \\
v_{\chi} v_\eta \lambda_5 & 2v_{\eta}^2 \lambda_{17} & \lambda_{\eta \xi} v_\eta v_\xi \\
\lambda_{\chi \xi} v_{\chi} v_\xi & \lambda_{\eta \xi} v_\eta v_\xi & 2\lambda_{\xi} v_{\xi}^2
\end{pmatrix}.
\]

Let us assume a simplified worth to be considered scenario which is characterized by the following relations:

\[
\lambda_5 = \lambda_{13} = \lambda_{17} = \lambda_{\xi} = \lambda_{\eta \xi} = \lambda, \quad v_\xi = v_{\chi}.
\]

The system of Eqs.(48 - 53) leads to another constraint, namely

\[
\sqrt{\lambda} > 0.
\]

In this scenario, the squared matrix (49) for the electrically neutral CP even scalars in the basis \( (R_{\eta}, R_{\chi}, R_{\xi}) \) takes the simple form:

\[
m_{\text{Peven3}}^2 = \lambda \begin{pmatrix}
2x^2 & x & x \\
x & 2 & 1 \\
x & 1 & 2
\end{pmatrix} v_{\chi}^2, \quad x = \frac{v_\eta}{v_{\chi}}.
\]
In this scenario, we find the that the physical scalars included in the matrix $m_{CPeven}^2$ are:

$$\begin{pmatrix}
  h \\
  H_4 \\
  H_5
\end{pmatrix} \simeq \begin{pmatrix}
  -1 + \frac{x^2}{3} & \frac{x}{3} & \frac{x}{3} \\
  0 & -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} \\
  \frac{\sqrt{x}}{3} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}}
\end{pmatrix}
\begin{pmatrix}
  R_\eta \\
  R_\chi \\
  R_{\xi 0}
\end{pmatrix},$$

(60)

where $h$ is the 126 GeV SM like Higgs boson. Thus, we find that the SM-like Higgs boson $h$ has couplings very close to SM expectation with small deviations of the order of $\frac{v_\chi}{v_x}$. In addition, the squared masses of the physical scalars included in the matrix $m_{CPeven}^2$ take the form:

$$m_h^2 \simeq \frac{4}{3} \lambda v_\eta^2, \quad m_{H_4}^2 \simeq \lambda v_\chi^2, \quad m_{H_5}^2 \simeq 3\lambda v_\chi^2.$$  

(61)

Taking into account the fact that mass of the SM Higgs boson is equal to 126 GeV, from (61) we obtain

$$\lambda \approx 0.187.$$  

(62)

Combining with the limit from the rho parameter in Ref. [16] $3.57 \text{TeV} \leq v_\chi \leq 6.9 \text{TeV}$ yields

$$1.5 \text{TeV} \leq m_{H_4} \leq 2.61 \text{TeV}, \quad 2.6 \text{TeV} \leq m_{H_5} \leq 4.5 \text{TeV}.$$  

(63)

### III.1.2. The charged Higgs bosons

The charged scalar sector contains two massless fields: $G_{W^+}$ and $G_{Y^+}$ which are Goldstone bosons eaten by the longitudinal components of the $W^+$ and $Y^+$ gauge bosons, respectively. The other massive fields are $\phi_1^+, \phi_2^+$ and $\phi_4^+$ with respective masses given in (18).

In the basis $(\rho_1^+, \rho_2^+, \phi_3^+)$, the squared mass matrix is given in (17). Let us make effort to simplify this matrix. Note that $\mu_{\phi_4}^2$, $\mu_{\phi_2}^2$, and $\mu_{\phi_3}^2$ can be derived using relations (14) and (57). In addition, it is reasonable to assume

$$\mu_p^2 = -\frac{v_\chi^2}{2} (\lambda_{18} + \lambda_\rho_5) \approx \mu_\eta^2, \quad \mu_{\phi_3}^2 = -\frac{v_\chi^2}{2} (\lambda_2^{\phi_3} + \lambda_\phi^{\phi_3}),$$

(64)

we obtain the simple form of the squared mass matrix of the charged Higgs bosons,

$$M_{chargeds}^2 = \begin{pmatrix}
  A + \frac{1}{2} v_\eta^2 (\lambda_6 + \lambda_9) & 0 & \frac{\lambda_2}{2} v_\eta v_\chi \\
  0 & \frac{1}{2} \left(v_\chi^2 \lambda_7 + \lambda_6 v_\eta^2\right) & \frac{1}{2} v_\eta v_\chi w_2 \\
  \frac{\lambda_2}{2} v_\eta v_\chi & \frac{1}{2} v_\eta v_\chi w_2 & \frac{1}{2} v_\eta^2 \lambda_2 \phi_3 \\
\end{pmatrix}.$$  

(65)

The matrix (65) predicts that there may exist two light charged Higgs bosons $H_{1,2}^+$ with masses at the electroweak scale and the mass of $H_3^+$ which is mainly composed of $\rho_3^+$ is around 3.5 TeV. In addition, the Higgs boson $H_1^+$ almost does not carry lepton number, whereas the others two do.

Generally, the Higgs potential always contains mass terms which mix VEVs. However, these terms must be small enough to avoid high order divergences (for examples, see Refs. [17,18]) and provide baryon asymmetry of Universe by the strong electroweak phase transition (EWPT).
Ignoring the mixing terms containing $\lambda_3$ in (65) does not affect other physical aspects, since the above mentioned terms just increase or decrease small amount of the charged Higgs bosons. Therefore, without lose of generality, neglecting the terms with $\lambda_3$ satisfies other aims such as EWPT.

Hence, in the matrix of (65), the coefficient $\lambda_3$ is reasonably assumed to be zero. Therefore we get immediately one physical field $\rho_1^+$ with mass given by

$$m_{\rho_1}^2 = \frac{1}{2} v_\eta^2 (\lambda_6 + \lambda_9).$$  \hfill (66)

The other fields mix by submatrix given at the bottom of (65). The limit $\rho_1^+ = H_1^+$ when $\lambda_3 = 0$ is very interesting for discussion of the Higgs contribution to the $\rho$ parameter.

Analysis of electroweak phase transition shows that the term of VEV mixing at the top-right corner should be negligible [17, 18] or $\lambda_3 \simeq 0$. (67)

Therefore, from (17), it follows that $\rho_1^+$ is physical field with mass

$$m_{\rho_1}^2 = A + \frac{1}{2} v_\eta^2 (\lambda_6 + \lambda_9),$$  \hfill (68)

and two massive bilepton scalars $\rho_3^+$ and $\phi_3^+$ mix each other by matrix at the right-bottom corner. Taking into account the conditions in (4) yields

$$A + \frac{1}{2} \left( v_\eta^2 \lambda_7 + v_\chi^2 \lambda_6 \right) > 0, \mu_{\phi_3}^2 + B_3 > 0,$$  \hfill (69)

$$\frac{1}{\sqrt{2}} v_\chi w_2 + \sqrt{\left( A + \frac{1}{2} \left( v_\eta^2 \lambda_7 + v_\chi^2 \lambda_6 \right) \right) \left( \mu_{\phi_3}^2 + B_3 \right)} > 0.$$  \hfill (70)

From (70) it follows that if $w_2 < 0$, then

$$\left( A + \frac{1}{2} \left( v_\eta^2 \lambda_7 + v_\chi^2 \lambda_6 \right) \right) \left( \mu_{\phi_3}^2 + B_3 \right) > \frac{v_\chi w_2^2}{2},$$

but if $w_2 > 0$, there are only conditions in (69).

It is worth mentioning that the masses of three charged scalars $\phi_i^+, i = 1, 2, 4$ are still not fixed.

Let us deal with the charged Higgs boson sector by assuming

$$\lambda_6 = \lambda_7 = \lambda_9 = \lambda_{18} = \lambda_3^\chi = \lambda_3^\eta = \lambda_3^{\phi_3} = \lambda'. \hfill (71)$$

With this assumption, we have

$$\mu_\chi^2 = - \frac{\lambda}{2} (3v_\chi^2 + v_\eta^2) \simeq - \frac{3}{2} \lambda v_\chi^2,$$

$$\mu_\eta^2 = - \lambda (v_\eta^2 + v_\chi^2) \simeq - \lambda v_\chi^2,$$  \hfill (72)

$$\mu_\xi^2 = - \frac{\lambda}{2} (v_\chi^2 + v_\eta^2) \simeq - \frac{1}{2} \lambda v_\chi^2.$$
In the basis \((\rho_1^+, \rho_3^+, \phi_3^+)\), the matrix in (17) becomes
\[
M_{\text{charged}}^2 = \begin{pmatrix}
\mu_\rho^2 + \lambda' v_\rho^2 + \lambda' v_\phi^2 & 0 & \frac{\lambda' v_\eta v_\chi}{2} \\
0 & \mu_\rho^2 + \frac{\lambda' v_\chi^2 + v_\eta^2}{2} & \frac{1}{\sqrt{2}} v_\chi w_2 \\
\frac{\lambda' v_\eta v_\chi}{2} & \frac{1}{\sqrt{2}} v_\chi w_2 & \mu_\phi^2 + \lambda_\phi^2 + \frac{1}{2} v_\eta^2
\end{pmatrix}.
\] (73)

Next, assuming
\[
\mu_\rho^2 = \mu_\phi^2 = \mu_\eta^2 = -\lambda' v_\chi^2,
\] (74)
we obtain
\[
M_{\text{new charged}}^2 = \begin{pmatrix}
\frac{\lambda' v_\chi^2}{2} & 0 & \frac{\lambda' v_\chi}{2} x \\
0 & \frac{\lambda' v_\chi^2 + v_\eta^2}{2} & \frac{1}{\sqrt{2}} v_\chi w_2 \\
\frac{\lambda' v_\chi}{2} x & \frac{1}{\sqrt{2}} v_\chi w_2 & \frac{\lambda_\phi^2 v_\eta^2}{2}
\end{pmatrix} v_\chi^2.
\] (75)

From (75), we get two charged Higgs bosons with masses at electroweak scale and one massive with mass around TeV \((\propto v_\chi)\), in addition the Higgs boson composed mainly from \(\rho_1^+\) does not carry lepton number, while the two others do.

**IV. CONCLUSION**

In this paper, we have applied the positivity of scalar mass spectra in the 3-3-1 model with CKS mechanism. We show that for the Higgs squared mass matrices, the conditions for positivity of the diagonal elements are most important since other constraints are followed from the first ones. In the model under consideration, the above conclusion is very helpful for the fixing parameters.

Since there are a lot of Higgs fields in this model, so the vacuum stability will be considered in the future study.

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